Field Theory and Statistical Hydrodynamics

The First Analytical Predictions of Anomalous Scaling

Misha Chertkov

Field theory is the most advanced subfield of theoretical physics that has been actively developing in the last 50 years. Traditionally, field theory is viewed as a formalism for solving many-body quantum mechanical problems, but the path or functional-integral representation of field theory is very useful in a much broader context. Here we discuss its use for predicting from first principles the stochastic, or turbulent, behavior of hydrodynamic flows.

The functional-integral formalism in field theory is a generalization of the famous Feynman-Kac path integral, which was introduced as a convenient alternative to the description of quantum mechanics by the Schrödinger equation. The path integral defines a quantum mechanical matrix element, or probability density for an observable, in terms of a sum over all possible trajectories, or variations, of the observable, some of which are forbidden by classical mechanics. In the more general functional-integral formalism, there is a field corresponding to each observable. In turn, to each field configuration, there is a corresponding statistical weight, and the product of the two is the integrand in the functional integral. The functional integral, which constitutes a summation or integration over many realizations of that field or observable, provides the probability distribution function for the observable.

In the 1950s, many researchers understood that any problem involving random variables, or random fields, can be interpreted in terms of a sum or integral over many field trajectories or field configurations. The simplest example is the problem of diffusion. There, the probability distribution function for the distance traveled by a single molecule as it collides at random with other molecules in a medium is calculated as a path integral over many Brownian motion trajectories. The integral reformulation is often advantageous because it allows one to utilize very powerful theoretical tools to evaluate or approximate the integrals. Perturbative analysis (often formulated in terms of diagrammatic techniques), saddle-point, or instanton, techniques, and various transformations (change of integration variables) are among the most useful tricks that allow analytical or semianalytical (numerical evaluation follows a theoretical step) evaluations.

Any problem in turbulence, or for that matter any statistical problem, formulated in terms of stochastic ordinary or partial differential equations can be reformulated in terms of a functional, or path, integral. Many researchers have contributed to the development of this field-theoretical approach to the problem of turbulent flow. This approach is now called statistical hydrodynamics. In the 1960s the most notable contributions to statistical hydrodynamics came from Robert Kraichnan (1967), who discovered the idea of the inverse cascade for two-dimensional (2-D) turbulence, and from Vladimir Zakharov (1967), who put the theory of wave turbulence on a firm mathematical ground by finding turbulence spectra as exact solutions and by introducing the notion of inverse and dual cascades in wave turbulence.

Perturbative (or diagrammatic) analysis, which was at the core of the Kraichnan-Zakharov analysis, defined the spirit of the most important theoretical results in statistical hydrodynamics for some 30 years following publication of Kraichnan and Zakharov's seminal papers cited above. The work described in those papers was cited in the award write-up for the 2003 Dirac Medal that went to Kraichnan and Zakharov.[1]

Between 1994 and 1995, however, three independent groups (refer to Chertkov et al. 1995, Chertkov and Falkovich 1996, Gawedzki and Kupiainen 1995, Shraiman and Siggia 1995, Pumir et al. 1997) came to the conclusion that the perturbative approach, which apparently led to self-similar scaling laws for the correlation and other structure functions of Navier-Stokes turbulence, did not work for passive scalar turbulence. By applying nonperturbative field-theoretic techniques, these groups were able to prove the existence of anomalous scaling in passive scalar turbulence. Below, we outline these new developments and discuss the possible implications for anomalous scaling in both theoretical and applied contexts of turbulent flow.

Intermittency and the Passive Scalar Model

Passive scalar turbulence describes the advection and diffusion of a scalar quantity (such as temperature or pollutant concentration) in a turbulent flow. The scalar quantity is described by a scalar field $\theta\,(t,\mathbf{r})$, and the dynamics of the scalar field evolve in space $\mathbf{r}$ and time t according to the following linear equation:

$$\partial_t \theta + (\mathbf{u}\nabla)\theta = \kappa\Delta\theta + \phi \quad , \tag{1}$$

where κ, $\mathbf{u}(t,\mathbf{r})$ and $\phi\,(t,\mathbf{r})$ stand for the diffusion (either thermal or material) coefficient, the incompressible velocity field, and the source field controlling injection of the scalar θ, respectively. The advection of θ is a passive process under the assumption that all three fields—velocity $\mathbf{u}$, injection ϕ, and scalar θ—are statistically independent of each other. That assumption, which is realistic in many practical cases, means that effects of the scalar field fluctuations on the flow (for example, buoyancy) are neglected.

A few years after Kolmogorov (1941) proposed the inertial cascade four-fifths law, relating third moment of velocity increment to the energy flux and energy dissipation in Navier-Stokes turbulence, Obukhov (1949) and Corrsin (1951) independently suggested that a similar consideration applies to the passive scalar problem. Indeed, if diffusion and injection are removed from Equation (1), then the integral of θ^2 over all space, $\int d\mathbf{r}\theta^2$, is conserved (or does not change with time). One can therefore consider θ^2 in the passive scalar problem as analogous to kinetic energy density, or $\mathbf{u}^2$, in Navier-Stokes turbulence. In any turbulent flow, the velocity fluctuations grow with scale size in the inertial range of scales, which lies between the dissipation scale η and the large forcing scaling L. Analogously, if the diffusion coefficient κ is small while the source field ϕ injects the "scalar energy" at a relatively large scale, L_ϕ, then advection dominates diffusion in the so-called convection range, which extends from L_ϕ down to the diffusive scale, r_d. The ratio of the two scales L_ϕ/r_d is a large dimensionless number that plays a role in passive scalar turbulence analogous to the role of the pumping-to-viscous scale ratio in the Navier-Stokes turbulence. That is, when the dimensionless ratio L_ϕ/r_d (closely related to the Peclet/Schmidt numbers) becomes large, passive scalar turbulence develops.

In the Obukhov-Corrsin picture of the passive scalar problem, once a large blob of the scalar field (that is, large on the scale of L_ϕ) is injected into a turbulent flow, turbulent advection causes a fine spatial structure of scalar inhomogeneities to develop within the initially homogeneous cloud. The finest scale of the spatial inhomogeneities is r_d because inhomogeneities at even smaller scales are smeared out by diffusion. In the language of the θ^2-energy "budget," the scalar energy density θ^2, which is permanently supplied at the large scale L_ϕ, cascades toward smaller scales within the convective range and is dissipated at the small scales, approximately r_d. Thus, the analog of Kolmogorov's four-fifths law for the scalar energy flux in passive scalar turbulence reads

$$\left\langle \theta_1 \mathbf{u}_2 \theta_2 \right\rangle = -\varepsilon_\phi r_{12} \quad , \tag{2}$$

where $\langle \ldots \rangle$ describes averaging, with respect to statistics, of both velocity and injection fields and ε_ϕ is the averaged scalar-energy dissipation rate, $\varepsilon_\phi = \kappa \langle (\nabla\theta)^2 \rangle$. In this Obukhov-Corrsin picture, the flux of θ^2 remains constant from scale to scale within the convective range, and the scalar-energy dissipation rate is equal to the scalar-energy input rate at the injection scale, estimated as $\varepsilon_\phi \sim \theta^2_{L_\phi} u_{L_\phi}/L_\phi$, where u_{L_ϕ} and θ_{L_ϕ} are typical values of velocity and scalar fluctuations at the injection scale.

Equation (2), which is the passive scalar analog of the four-fifths law controlling the scalar energy budget, is exact. The exact statement, however, is limited to the very special correlation function, and no generalization is known of Equation (2) for other simultaneous correlation functions of the scalar field. This caveat was "fixed" by Obukhov and Corrsin, who conjectured self-similarity of scalar fluctuations. The conjecture is akin to Kolmogorov's self-similarity assumption for velocity fluctuations.

The self-similarity for the scalar-field statistics looks simple and thus appealing. However, accurate experimental measurements between the 1960s and the 1980s (Sreenivasan 1991, Sreenivasan and Antonia 1997), supported neither the Kolomogorov nor the Obukhov-Corrsin predictions for self-similar scaling laws, thus offering an early hint that anomalous scaling is common in turbulence. For the passive scalar increments, the anomalous scaling scenario means that the moments of scalar increments have the following form:

$$\left\langle \left[\theta(r) - \theta(r+l)\right]^{2n} \right\rangle \sim \frac{\varepsilon_\phi^n}{\varepsilon^{n/3}} \, l^{4n/3} \left(\frac{L_\phi}{l}\right)^{\Delta_{2n}} \quad , \tag{3}$$

where $\Delta_{2n} > 0$ is the anomalous exponent. In this formal description, the self-similar scenario would correspond to $\Delta_{2n} = 0$. The anomalous scaling, and thus lack of self-similarity, appeared to be much more pronounced in the experimental data for the scalar field than for the velocity field. Because at that time there was no theoretical understanding of the origin of anomalous scaling, the observations were essentially rejected as spurious.

Resolution of the standoff on anomalous scaling emerged in the mid-1990s. First, Kraichnan proposed (1994) an ad hoc scheme for producing a closed set of equations for what is today called the Kraichnan model. This microscopic model, initially introduced in 1967 (refer to Kraichnan 1967), deals with passive scalar turbulence for a velocity field in Equation (1) that has self-similar statistics. The velocity field in the model was chosen to be incompressible, Gaussian, and short correlated (δ-correlated) in time. Spatial correlations in the model are characterized by the pair correlation function of the velocity difference between two points measured at two distinct times:

$$\left\langle \left(v^{\alpha}(t_1;r_1) - v^{\alpha}(t_1;r_2)\right)\left(v^{\beta}(t_2;r_1) - v^{\beta}(t_2;r_2)\right)\right\rangle = \delta(t_1 - t_2) K^{\alpha\beta}(r_1 - r_2) \tag{4}$$

where $\alpha, \beta = 1, \cdots, d$. The eddy diffusivity tensor $K^{\alpha\beta}(r)$ is growing algebraically with the spatial separation $K \propto r^{2-\gamma}$ so that the exponent characterizing the degree of non-smoothness of the synthetic velocity field γ and the spatial dimensionality d are two independently controlled parameters. In his 1994 paper, Kraichnan proposed an approximate closure scheme resulting in a closed set of equations for scalar structure functions of order 4, $S_4(l) = \langle[\theta(\mathbf{r} + \mathbf{1}) - \theta(\mathbf{r})]^4\rangle$, and higher. The main message here was that, although the velocity field exhibited self-similarity, the scalar fluctuations are extremely intermittent and thus characterized by an anomalous expression generalizing Equation (3)

$$\left\langle \left[\left(\theta(r+l)\right) - \theta(r)\right]^{2n} \right\rangle \sim l^{n\xi_2} \left(\frac{L_\phi}{l}\right)^{\Delta_{2n}} \propto l^{\xi_{2n}} \tag{5}$$

with $\xi_{2n} \neq n\xi_2$ and $\Delta_{2n} \neq 0$. Then, independently, and almost simultaneously, three groups (refer to Chertkov et al. 1995, Chertkov and Falkovich 1996, Gawedzki and Kupiainen 1995, Shraiman and Sigia 1995, Pumir et al. 1997) developed a rather different approach that required no ad hoc closure assumptions.

The new approach focused on the analysis of the simultaneous correlation function of the scalar field taken at four different points, $F_{1234} \equiv \langle\theta(t,\mathbf{r}_1)\,\theta(t,\mathbf{r}_2)\,\theta(t,\mathbf{r}_3)\,\theta(t,\mathbf{r}_4)\rangle$. That four-point correlation function is governed by a second-order linear, and therefore closed (!!!), partial differential inhomogeneous equation,

$$\hat{L}F_{1234} = \chi\ , \tag{6}$$

where

$$\hat{L} \equiv \sum_{i,j} K^{\alpha\beta}(r_i - r_j)\nabla_i^{\alpha}\nabla_j^{\beta}$$

is the differential operator of the second order, called eddy diffusivity operator, and χ is a known function, so that no ad hoc closure was required. The solution of any linear differential equation can be presented for a subinternal range of scales as a sum of homogeneous and certain inhomogeneous solutions of the equation. (For the four-point correlation function, the subinternal range would be the convective range of scales in which the separations between the four points are larger than the diffusive scale but smaller than the scalar injection scale.) Progress came from the recognition that the anomalous scaling contributions to the four- through n-point correlation functions, and respectively to the fourth- through nth-order moments of the scalar increments (that is, structure functions), originate primarily from homogeneous solutions of the partial differential equation, that is, from the zero modes Z of the eddy diffusivity operator $LZ = 0$. Thus, the first important outcome of the analysis was that the value of the anomalous exponent for the passive scalar structure functions was insensitive to the strength of the forcing field. It was also shown that the anomalous contribution originates from matching the homogeneous and inhomogeneous solutions at the injection scale rather than the diffusive scale. Zero modes of the eddy diffusivity operator were analyzed and anomalous corrections Δ_{2n} were calculated in some important limits of the Kraichnan model corresponding to (a) a high spatial dimension, $d \to \infty$, so that calculations were done in an expansion with respect to $1/d$ (Chertkov et al 1995, Chertkov and Falkovich 1996), (b) an extremely irregular (diffusive) velocity, $2 - \gamma << 1$ (Gawedzki and Kupiainen 1996), (c) an almost spatially smooth velocity, $\gamma << 1$ (Shraiman and Siggia 1995, Pumir et al. 1997), and later for (d) a large deviation, or instanton, regime for which it was shown that the structure function exponent ξ_{2n} saturates to a constant (Chertkov 1997, Balkovsky and Lebedev 1998). For the first time ever, analytical calculations of a turbulence problem predicted the existence and the degree of anomalous scaling.

Passive transport in general and anomalous scaling in particular have also been given a transparent Lagrangian interpretation: It was shown that the n-point Eulerian (simultaneous) correlation function can be reinterpreted in terms of Lagrangian trajectories of n particles/markers evolving in the same velocity field. Thus, the Eulerian pair-correlation function of the scalar field $\langle \theta(\mathbf{r} + \mathbf{1})\theta(\mathbf{r})\rangle$ is equal to the value of the θ^2 energy flux ε_ϕ multiplied by the time $\langle T_{l \to L_\phi}\rangle$, which is defined as the average (over velocity field statistics) of the time for two particles released a distance r_{12} apart to become separated by a distance L_ϕ. In this Lagrangian interpretation, the anomalous scaling is related to correlations between Lagrangian trajectories of different particles—for example, $\langle T_{r_{12}\to L} T_{r_{34}\to L}\rangle \neq \langle T_{r_{12}\to L}\rangle\langle T_{r_{34}\to L}\rangle$. (That is, two pairs of particles, 1-2 and 3-4 respectively, released in the same flow diverge so that both r_{12} and r_{34} reach the integral scale L in finite times, $T_{r_{12}\to L}$ and $T_{r_{34}\to L}$, respectively. However, if the gedanken experiment is repeated many times, one finds that the two times are actually correlated; that is, they are statistically dependent. The Lagrangian interpretation of passive scalar transport has also allowed efficient numerical analysis of the problem (Frisch et al. 1998), leading to accurate validation of the theoretical results but, more important, to a wide exploration of anomalous scaling in the intermediate parametric region—away from the asymptotic limits considered in Chertkov et al. (1995), Chertkov and Falkovich (1996), Gawedski and Kupiainen (1995), Shraiman and Siggia (1995), Pumir et al. (1997), Chertkov (1997) and Balkovsky and Lebedev (1998)—where quantitative theoretical analysis had been hopeless.

In an independent development, Burgers turbulence (or simply "Burgulence") was found to have anomalous scaling of an extreme kind: The left (negative) values' tail of the probability distribution function for the velocity increment is of extremely extended, algebraic form (Chekhlov and Yakhot 1995, Polyakov 1995, Khanin et al. 1997, Frisch and Bec 2001).

These nonperturbative results on anomalous scaling in relatively simple problems are recognized as the most important breakthrough in the theory of turbulence for the following reasons: (1) They prove that anomalous scaling as an extreme form of intermittency does exist. They also demonstrate that anomalous scaling is a generic phenomenon. Now, rather than proving the existence of anomalous scaling, the major task is to explain why the anomalous scaling exponent is so small (although still distinguishable from zero) in many more complex situations such as isotropic homogeneous Navier-Stokes turbulence. (2) The new nonperturbative approach has benefited from a Lagrangian description. Thus, in the passive scalar case, differential equations for scalar correlation functions can be reinterpreted in terms of a path integral over many Lagrangian trajectories (each set of trajectories corresponding to a single realization of velocity field). (3) The development of scalar turbulence theory (Shraiman and Siggia 2000, Falkovich et al 2001) has also generated new results in related areas of research such as kinematic dynamo theory (Vergassola 1996, Chertkov et al. 1999), enhancement of chemical reactions by turbulence (Chertkov 1999, Chertkov and Lebedev 2003), polymer stretching by turbulence (Balkovsky et al. 2000 and 2001; Chertkov 2000), elastic turbulence (Fouxon and Lebedev 2003), and more.

The progress achieved in scalar turbulence has also generated a resurgence of interest in more complex problems in statistical hydrodynamics. Motivated by the Lagrangian representation of passive scalar transport, we and colleagues have found a finite number of Lagrangian particles (four, or a tetrad, is the minimum number—see Chertkov et al. (1999) can be considered a sensible closure framework for a Lagrangian phenomenological model of Navier-Stokes turbulence. Finally, the two solvable models have opened possibilities for benchmarking various nonperturbative methods of statistical hydrodynamics such as instanton calculus (Chertkov 1997, Balkovsky and Lebedev 1998, Falkovich et al. 1996, Balkovsky et al. 1997). Our optimistic expectation is that these powerful theoretical methods may soon deliver new results for more complex and challenging problems in statistical hydrodynamics, including homogeneous isotropic Navier-Stokes turbulence, shear-driven turbulence, and perhaps even Rayleigh-Taylor turbulent mixing and magnetohydrodynamic turbulence. ■

Further Reading

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For further information, contact Misha Chertkov (505) 665-8119 (chertkov@lanl.gov).